Journal of Statistical Physics, Vol. 12, No. 2, 1975

A Functional Phase-Integral Method and Applications to the Laser Beam Propagation in Random Media

Pao-Liu Chow[1]

Received July 15, 1974

The problem of propagation of a high-intensity light beam in a half-space with random inhomogeneities is treated. An exact solution is constructed through a functional integral representation. For a Gaussian random field, the exact moments of solution are given explicitly. A functional phase-integral method is developed to provide an asymptotic evaluation of the moment integrals. The method is applied to two problems in a stochastic laser beam propagation in random media with a homogeneous background or with a focusing effect.

KEY WORDS: Laser; random media; functional integral; asymptotics.

1. INTRODUCTION

In an earlier paper,[1] hereafter referred to as I, we showed how the method of function space (or functional) integration can be applied effectively to certain

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This work was supported in part by United States Army Research Office–Durham.

[1] Department of Mathematics, Wayne State University, Detroit, Michigan.

problems in wave propagation in an unbounded random medium. To justify the smoothing perturbation technique and the direct interaction formalism discussed in I, we treated rigorously a random parabolic equation,[2] where error bounds for these approximations were obtained. Here we shall extend an asymptotic method, known as the phase-integral method or the method of stationary phase, presented in I to a half-space problem, and then apply it to the propagation of a laser beam through a turbulent medium.

For this purpose, let us consider the time-harmonic wave propagation of a focused beam of the high-intensity light in the half-space $x > 0$, where the wave function u satisfies the reduced wave equation

$$\Delta u(\mathbf{r}, \omega) + k^2 n^2(\mathbf{r}, \omega) u(\mathbf{r}, \omega) = 0, \qquad x > 0 \tag{1}$$

Here Δ denotes the Laplacian operator in the three-dimensional space variable $\mathbf{r}$; x is the first component of $\mathbf{r}$; k is a complex wave number with a positive imaginary part; and $n(\mathbf{r}, \omega)$ is the random refractive index, which is a random function of $\mathbf{r}$, with ω in a sample space Ω. We assume that the wave function u represents the propagation of light due to a source emitted by a transmitting laser at $x = 0$, so that

$$u(\mathbf{r})|_{x=0} = u_0(\boldsymbol{\rho}) = A(\boldsymbol{\rho}) \exp[ik\phi(\boldsymbol{\rho})] \tag{2}$$

where A and ϕ are the prescribed amplitude and phase of the light beam at $x = 0$, and $\boldsymbol{\rho}$ is the transverse variable of $\mathbf{r} = (x, \boldsymbol{\rho})$. Here and hereafter the dependence of u on ω is often omitted when there is no confusion.

The main objective of this paper is to determine the moments of the solution to the random equation (1) subject to the boundary condition (2) and a radiation condition at $|\mathbf{r}| = \infty$, $x > 0$, which will not be written down. We shall show that this problem can be solved exactly in terms of Wiener integrals, which are then evaluated asymptotically for large k. When specified to special cases, they yield, among others, some known results obtained by different approaches.[3,4] As a by-product, we found that the parabolic equation approximation used in high-frequency wave propagation corresponds to a unidirectional asymptotic expansion, as shown in the appendix.

A systematic study on stochastic laser beam propagation was first made by Schmeltzer,[5] using the Rytov method or the logarithmic regular perturbation method.[6] De Wolf[3] tried to solve this problem by a combination of geometric optics and selected summation of perturbation series. By assuming n^2 to be the product of a random function of x and quadratic in $\boldsymbol{\rho}$, the beam problem was analyzed by Papanicolaou *et al.*[7] after a parabolic equation approximation.

2. CONSTRUCTION OF EXACT SOLUTION

To solve the problem (1)–(2) exactly, we extend the random function $n(\mathbf{r}, \omega)$ symmetrically with respect to x. Let $\hat{n}(\mathbf{r}, \omega)$ be the extension defined as

$$\begin{aligned} \hat{n}(\mathbf{r}, \omega) \equiv \hat{n}(x, \boldsymbol{\rho}, \omega) &= n(x, \boldsymbol{\rho}, \omega), \qquad x \geqslant 0 \\ &= n(-x, \boldsymbol{\rho}, \omega), \qquad x \leqslant 0 \end{aligned} \tag{3}$$

Consider the full space, random Green's function $G(\mathbf{r}, \mathbf{r}', \omega)$ which satisfies the following equation:

$$\Delta G(\mathbf{r}, \mathbf{r}', \omega) + k^2 \hat{n}^2(\mathbf{r}, \omega) G(\mathbf{r}, \mathbf{r}', \omega) = \delta(\mathbf{r} - \mathbf{r}') \tag{4}$$

where $\delta(\mathbf{r})$ stands for the Dirac delta function, and G is outgoing at $|\mathbf{r}| = \infty$. Then it is well known that, by a reflection principle (or method of image), the solution to the half-space problem (1)–(2) can be represented in terms of the full space Green's function according to

$$u(\mathbf{r}) = -2(\partial/\partial x) \int_{R_2} G(\mathbf{r}, \boldsymbol{\rho}') u_0(\boldsymbol{\rho}')\, d\boldsymbol{\rho}' \tag{5}$$

in which u_0 is defined as in (2), and the integration is over the whole plane $x' = 0$.

As shown in I, the radiation problem (4) can be transformed into an initial value problem for a parabolic equation, and is thereby solved by a functional integration:

$$G(\mathbf{r}, \mathbf{r}') = (ik)^{-1} \int_0^\infty E_z\left\{ \exp\left[ik \int_0^t \hat{n}^2(\mathbf{z}(\tau) + \mathbf{r}')\, d\tau \right] \middle| \mathbf{z}(0) = 0, \; \mathbf{z}(t) = \mathbf{r} - \mathbf{r}' \right\} \psi(t, \mathbf{r} - \mathbf{r}')\, dt \tag{6}$$

where $\psi(t, \mathbf{r})$ is the complex heat kernel defined to be the principal branch of

$$\psi(t, \mathbf{r}) = (k/4\pi i t)^{3/2} \exp(ikr^2/4t) \tag{7}$$

and $E_z\{\cdot|\mathbf{z}(0) = 0, \mathbf{z}(t) = \mathbf{r}\}$ designates the conditional Wiener expectation with the complex variance parameter $2ik^{-1}$, given that the paths $\mathbf{z}(\tau)$ start from $\mathbf{z} = 0$ at $\tau = 0$ and reach $\mathbf{z} = \mathbf{r}$ at $\tau = t$. Alternatively, we may view it as a Gaussian integration over the set $C(t, \mathbf{r})$ of continuous functions $\mathbf{z}(\tau)$ on $[0, t]$ with $\mathbf{z}(0) = 0$, $\mathbf{z}(t) = \mathbf{r}$. Let $G_t[\mathbf{z}]$ be a smooth functional on $C(t, \mathbf{r})$.

Then, for computational convenience, it is desirable to introduce the symbolic expression, with $\dot{\mathbf{z}} = d\mathbf{z}/d\tau$,

$$E_z\{G_t[\mathbf{z}]|\mathbf{z}(0) = 0, \mathbf{z}(t) = \mathbf{r}\}\psi(t, \mathbf{r}) = \int_{C(t,\mathbf{r})} G_t[\mathbf{z}] \exp\left\{\tfrac{1}{4}ik \int_0^t [\dot{\mathbf{z}}(\tau)]^2 \, d\tau\right\} d_W\mathbf{z} \tag{8}$$

Now, in view of (5), (6), and (8), the exact random solution u can be written in the form

$$u(\mathbf{r}, \omega) = \frac{2i}{k}\frac{\partial}{\partial x}\int_{R_2}\int_0^\infty\int_{C(t,\mathbf{r}-\boldsymbol{\rho}')} u_0(\boldsymbol{\rho}') \times \exp\left\{ik\int_0^t \hat{n}^2(\mathbf{z}(\tau) + \boldsymbol{\rho}')\, d\tau + \frac{ik}{4}\int_0^t [\dot{\mathbf{z}}(\tau)]^2\, d\tau\right\} d\boldsymbol{\rho}'\, dt\, d_W\mathbf{z} \tag{9}$$

where $u_0(\boldsymbol{\rho})$ is given by (2).

To compute the moments of u, we assume that

$$n^2(\mathbf{r}, \omega) = a(\mathbf{r}) + \epsilon\mu(\mathbf{r}, \omega), \qquad x \geqslant 0 \tag{10}$$

where $a(\mathbf{r})$ is the mean of n^2; $\mu(\mathbf{r}, \omega)$ is a centered Gaussian random field; and ϵ is a parameter with $0 < \epsilon \leqslant 1$. Let the angular bracket $\langle\cdot\rangle$ denote the mathematical expectation over Ω. Then we have $\langle n^2\rangle = a$, $\langle\mu\rangle = 0$, and the covariance function of μ is given by

$$\langle\mu(\mathbf{r}, \omega)\mu(\mathbf{r}', \omega)\rangle = R(\mathbf{r}, \mathbf{r}') \tag{11}$$

Noting (1), the extended random field $\hat{n}^2$ has a mean

$$\hat{a}(\mathbf{r}) \equiv \hat{a}(x, \boldsymbol{\rho}) = a(|x|, \boldsymbol{\rho}) \tag{12}$$

and the corresponding covariance function of $\hat{\mu}$ is

$$\hat{R}(\mathbf{r}, \mathbf{r}') \equiv \hat{R}(x, \boldsymbol{\rho}; x', \boldsymbol{\rho}') = R(|x|, \rho; |x'|, \boldsymbol{\rho}') \tag{13}$$

For $m = 1, 2, ..., n$, let us define the mth moment of u as follows:

$$\Gamma_m(\mathbf{r}, \mathbf{r}_2, ..., \mathbf{r}_m) = \langle u_1(\mathbf{r}_1, \omega)u_2(\mathbf{r}_2, \omega)\cdots u_m(\mathbf{r}_m, \omega)\rangle \tag{14}$$

where

$$\begin{aligned} u_j &= u \qquad \text{for odd } j \\ &= \bar{u} \qquad \text{for even } j \end{aligned} \tag{15}$$

and $\bar{u}$ means the complex conjugate of u.

Similar to our results in I, all moments Γ_m can be obtained explicitly for the Gaussian case. To this end, let us introduce the following abbreviations:

$$M_j(t_j) = \int_0^{t_j} \hat{a}(\mathbf{z}_j(\tau) + \boldsymbol{\rho}_j')\, d\tau, \qquad j = 1, 2, ..., m \tag{16}$$

$$S_{jl}(t_j, t_l) = \int_0^{t_j}\int_0^{t_l} \hat{R}(\mathbf{z}_j(\tau_1) + \boldsymbol{\rho}_j', \mathbf{z}_l(\tau_2) + \boldsymbol{\rho}_l')\, d\tau_1\, d\tau_2 \tag{17}$$

$$j, l = 1, 2, ..., m$$

Then, noting (9)–(17), it is not difficult to verify that,[1] for $m = 1, 2, ...,$

$$\begin{aligned}
&\Gamma_m(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_m) \\
&\quad = (-1)^{m+1}(2i)^m(k_1k_2\cdots k_m)^{-1} \frac{\partial^m}{\partial x_1\, \partial x_2 \cdots \partial x_m} \\
&\quad + \left(\int_{R_2}\int_0^\infty \int_{C_1}\right) \overset{(m)}{\cdots} \left(\int_{R_2}\int_0^\infty \int_{C_m}\right) u_{01}(\boldsymbol{\rho}_1') \cdots u_{0m}(\boldsymbol{\rho}_m') \\
&\quad \times \exp\left\{\sum_{j=1}^m i_jk_jM_j(t_j) - \tfrac{1}{2}\sum_{j=1}^m \beta_j^2 S_{jj}(t_j, t_j)\right. \\
&\quad \left. - \tfrac{1}{2}\sum_{\substack{j\neq l\\ j,l=1}}^m \beta_j\beta_l S_{jl}(t_j, t_l) + \tfrac{1}{4}\sum_{k=1}^m i_mk_m \int_0^{t_j} [\dot{\mathbf{z}}_j(\tau)]^2\, d\tau\right\} \\
&\quad \times (d\boldsymbol{\rho}_1'\, dt_1\, d_W\mathbf{z}_1)\cdots(d\boldsymbol{\rho}_m'\, dt_m\, d_W\mathbf{z}_m)
\end{aligned} \tag{18}$$

where $C_j = C(t_j, \mathbf{r}_j - \boldsymbol{\rho}_j')$, $\beta_j = k_j\epsilon$, and convention (15) applies to i, k, u_0, and ψ.

For an arbitrary random field μ, other than a Gaussian process, the exact moments can be expressed in terms of its characteristic functional, as shown in I. Also, we wish to point out that, for μ Gaussian, there is a finite probability that n^2 may become negative. However, the probability of the tail distribution is hopefully small, at least, when ϵ is small.

3. ASYMPTOTIC METHOD

In this section, we shall evaluate the moments Γ_m asymptotically as $k \to \infty$, with β held fixed. For $m = 1$, (18) reads

$$\Gamma_1(\mathbf{r}) = 2ik^{-1}(\partial/\partial x)J(\mathbf{r}) \tag{19}$$

where J is the integral

$$J(\mathbf{r}) = \int_{R_2} \int_0^\infty \int_{C(t,\mathbf{r}-\boldsymbol{\rho}')} u_0(\boldsymbol{\rho}') \times \exp\left\{ik\left[M(t) + \tfrac{1}{4}\int_0^t [\dot{\mathbf{z}}(\tau)]^2\, d\tau\right] - \tfrac{1}{2}\beta^2 S(t,t)\right\} \times d\boldsymbol{\rho}'\, dt\, d_W\mathbf{z} \tag{20}$$

In view of the definitions (16) and (17), the above integral J can be put in the form

$$J(\mathbf{r}) = \int_{R_2} \int_0^\infty u_0(\boldsymbol{\rho}') K(t, \mathbf{r}, \boldsymbol{\rho}')\, d\boldsymbol{\rho}'\, dt \tag{21}$$

Here

$$K(t, \mathbf{r}, \boldsymbol{\rho}') = \int_{C(t,\mathbf{r}-\boldsymbol{\rho}')} G_t[\mathbf{z}] \times \exp\left(ik \int_0^t \{\tfrac{1}{4}[\dot{\mathbf{z}}(\tau)]^2 + \hat{a}[\mathbf{z}(\tau) + \boldsymbol{\rho}']\}\, d\tau\right) d_W\mathbf{z} \tag{22}$$

and the functional G_t is defined as

$$G_t[\mathbf{z}] = \exp\left[-\tfrac{1}{2}\beta^2 \int_0^t \int_0^t \hat{R}(\mathbf{z}(\tau_1) + \boldsymbol{\rho}', \mathbf{z}(\tau_2) + \boldsymbol{\rho}')\, d\tau_1\, d\tau_2\right] \tag{23}$$

Now we apply the phase-integral method (or the Laplace method, in general) to (21) to obtain an asymptotic evaluation for large k. This will be carried out in two steps. The first step consists in seeking the extremal paths $\mathbf{z}^*(\tau)$ over the class $C(t, \mathbf{r} - \boldsymbol{\rho}')$ that render the exponent in (22) stationary. The variational problem for determining $\mathbf{z}^*$ yields the Euler equation

$$\frac{1}{2}\frac{d^2\mathbf{z}(\tau)}{d\tau^2} - \nabla\hat{a}(\mathbf{z}(\tau)) = 0, \qquad 0 \leqslant \tau \leqslant t$$
$$\mathbf{z}(0) = 0, \qquad \mathbf{z}(t) = \mathbf{r} - \boldsymbol{\rho}' \tag{24}$$

where ∇ is the gradient operator.

Suppose that the boundary-value problem (24) has a unique solution $\mathbf{z}^*(\tau)$. Then, approximating the phase functional in (22) by a quadratic functional about $\mathbf{z}^*$, we have, for large k,

$$K(t, \mathbf{r}, \boldsymbol{\rho}') \sim K_1(t, \mathbf{r}, \boldsymbol{\rho}') \tag{25}$$

and

$$K_1(t, \mathbf{r}, \boldsymbol{\rho}') = \exp\left(ik \int_0^t \{\tfrac{1}{4}[\dot{\mathbf{z}}^*(\tau)]^2 + \hat{a}[\mathbf{z}^*(\tau)]\}\, d\tau\right) \times \int_{C(t,0)} G_t[\mathbf{z}^* + \mathbf{y}] \exp\left[\tfrac{1}{2}ik \int_0^t \mathbf{y}(\tau)\cdot(\nabla\nabla\hat{a}[\mathbf{z}(\tau)])\cdot\mathbf{y}(\tau)\, d\tau + \tfrac{1}{4}ik \int_0^t [\dot{y}(\tau)]^2\, d\tau\right] d_W\mathbf{y} \tag{26}$$

in which $\mathbf{y} = \mathbf{z} - \mathbf{z}^*$, and $\nabla\nabla\hat{a}$, a second order tensor, denotes the second gradient of $\hat{a}$. If there exist more than one solution to (24), then (25) becomes the sum over all contributions due to the multiple solutions $\mathbf{z}^*$. When the mean function $a(\mathbf{r})$ is constant, (26) can be greatly simplified by evaluating $G_t[\mathbf{y}]$ at $\mathbf{z}^*$ to give

$$K_1(t, \mathbf{r}, \boldsymbol{\rho}') \sim G_t[\mathbf{z}^*]\psi(t, \mathbf{r} - \boldsymbol{\rho}') \exp(ikat) \tag{27}$$

The second step requires expressing K_1 in the polar form

$$K_1(t, \mathbf{r}, \boldsymbol{\rho}') = A_1(t, \mathbf{r}, \boldsymbol{\rho}') \exp\{ik\phi_1(t, \mathbf{r}, \boldsymbol{\rho}')\} \tag{28}$$

Then we use (28) and (22), which is in turn used in (21) to get

$$J(\mathbf{r}) \sim \int_{R_2} \int_0^\infty u_0(\boldsymbol{\rho}')A_1(t, \mathbf{r}, \boldsymbol{\rho}') \exp\{ik\phi_1(t, \mathbf{r}, \boldsymbol{\rho}')\}\, d\boldsymbol{\rho}'\, d\tau \tag{29}$$

Noting the expressions for u_0 and ψ given by (2) and (7), respectively, the integral (29) is of the Laplace type. Hence the conventional Laplace method is applicable here. To proceed, it is found convenient to first locate the stationary points along the t axis for a fixed $\boldsymbol{\rho}'$. They are the solutions $t^*(\mathbf{r}, \boldsymbol{\rho}')$ to the following equation:

$$\partial\phi_1(t, \mathbf{r}, \boldsymbol{\rho}')/\partial t = 0 \tag{30}$$

For abbreviation, let us set

$$\int_0^\infty \exp\{ik\phi_1(t, \mathbf{r}, \boldsymbol{\rho}')\}\, dt = A_2(\mathbf{r}, \boldsymbol{\rho}') \exp\{ik\phi_2(\mathbf{r}, \boldsymbol{\rho}')\} \tag{31}$$

and

$$A_3(\mathbf{r}, \boldsymbol{\rho}') = A(\boldsymbol{\rho}')A_1(t^*(\mathbf{r}, \boldsymbol{\rho}'), \mathbf{r}, \boldsymbol{\rho}')A_2(\mathbf{r}, \boldsymbol{\rho}') \tag{32}$$

Then, taking (2) and (27)–(32) into account and performing the t-integration, (29) becomes

$$J(\mathbf{r}) \sim \int_{R_2} A_3(\mathbf{r}, \boldsymbol{\rho}') \exp\{ik[\phi(\boldsymbol{\rho}') + \phi_2(\mathbf{r}, \boldsymbol{\rho}')]\}\, d\boldsymbol{\rho}' \tag{33}$$

It follows from (19) that

$$\Gamma_1(\mathbf{r}) \sim 2ik^{-1}\int_{R_3}\left\{\frac{\partial A_3(\mathbf{r},\boldsymbol{\rho}')}{\partial x} + ikA_3(\mathbf{r},\boldsymbol{\rho}')\frac{\partial\phi_2(\mathbf{r},\boldsymbol{\rho}')}{\partial x}\right\}$$

$$+ \exp\{ik[\phi(\boldsymbol{\rho}') + \phi_2(\mathbf{r},\boldsymbol{\rho}')]\}\, d\boldsymbol{\rho}' \tag{34}$$

Therefore the total phase of the above integral is $\phi + \phi_2$, which is stationary when

$$\nabla'\phi(\boldsymbol{\rho}') + \nabla'\phi_2(\mathbf{r},\boldsymbol{\rho}') = 0 \tag{35}$$

Here ∇' denotes the gradient operator in $\boldsymbol{\rho}'$. Let $\boldsymbol{\rho}' = \boldsymbol{\rho}^*(\mathbf{r})$ be a solution of (35) and let, for $j, l = 1, 2$,

$$d_{jl}(\mathbf{r}) = \frac{\partial}{\partial\rho_j'\,\partial\rho_l'}\,[\phi(\boldsymbol{\rho}') + \phi_2(\mathbf{r},\boldsymbol{\rho}')]_{\boldsymbol{\rho}'=\boldsymbol{\rho}^*(\mathbf{r})} \tag{36}$$

Then, corresponding to each $\boldsymbol{\rho}^*$, the main contribution to the integral (36) comes from the neighborhood of ρ^* so that

$$\Gamma_1(\mathbf{r}) \sim 2ik^{-1}A_3(\mathbf{r},\boldsymbol{\rho}^*(\mathbf{r}))\exp\{ik[\phi(\boldsymbol{\rho}^*(\mathbf{r})) + \phi_2(\mathbf{r},\boldsymbol{\rho}^*(\mathbf{r}))]\}$$

$$\times\int_{R_2}\exp\left\{\tfrac{1}{2}ik\sum_{l,m=1}^{2} d_{lm}(\mathbf{r})\rho_l'\rho_m'\right\} d\boldsymbol{\rho}'$$

$$= \frac{2}{|\det D(\mathbf{r})|^{1/2}}\,A_3(\mathbf{r},\boldsymbol{\rho}^*(\mathbf{r}))\exp\{ik[\phi(\boldsymbol{\rho}^*(\mathbf{r})) + \phi_2(\mathbf{r},\boldsymbol{\rho}^*(\mathbf{r}))]\} \tag{37}$$

where $D(\mathbf{r})$ is the diagonalized form of the symmetric matrix $[d_{ij}(\mathbf{r})]$, and det D stands for the determinant of D, which is assumed to be nonzero. The singular points at which det $D(\mathbf{r})$ vanishes correspond to the caustics for the mean wave. This completes the asymptotic evaluation of the first moment Γ_1.

For higher moments $(m > 1)$ under the same limits, $k \to \infty$ with $\beta = k\epsilon$ held fixed, the computations turn out to be similar to the case for $m = 1$. Since the m-iterated integral (20) with respect to t, $\boldsymbol{\rho}'$, and $\mathbf{z}(\tau)$ are not coupled through the mean field $a(\mathbf{r})$, the asymptotic evaluation can be done by treating the m-fold integrations independently. The stationary path $\mathbf{z}_j^*(\tau)$, $j = 1, 2, \ldots, m$, corresponding to the jth integration is determined in exactly the same way as the case for $m = 1$ indicated above.

However, the asymptotic evaluation becomes much more complicated under the different limits when $k \to \infty$, $k\epsilon^2 = O(1)$ $[\beta = O(k)]$ or when $k \to \infty$, $\beta = k^2$ $(\epsilon = 1)$. Then the Euler equation (24) becomes a nonlinear integrodifferential equation involving the covariance R (see Ref. 1) and there exist no real solutions. This gives rise to a difficult mathematical question: Can we deform the "path" of integration in function space from the class $C(t, \mathbf{r})$ to a certain class of complex functions containing the complex stationary paths $\mathbf{z}^*(\tau)$ and $C(t, \mathbf{r})$, similar to the finite-dimensional case, and how should it be done? Another puzzle is this. For $\epsilon = 1$ we can apply the asymptotic method to the random solution (9). The random Euler equation, similar to the result of the geometric optics approximation, admits a real, random solution (see Ref. 6). Since the asymptotic approximations executed before and after taking the expectation of the random solution seem to yield different results for moments, the question is which is physically correct and why? In view of the above difficulties, the asymptotic evaluation for a large k and a fixed β seems to be the only feasible one to facilitate the actual computations.

4. APPLICATIONS TO LASER BEAM PROBLEMS

Let us apply the phase-integral method developed in Section 3 to two problems in laser beam propagation in turbulent media. The first problem is concerned with propagation of a laser light through a random medium with a homogeneous background, such as the turbulent atmosphere,[4] and the second problem pertains to its propagation in a focusing medium with random inhomogeneities, such as a hot gas lens.[8,9]

4.1. Homogeneous Random Media

In this case, the mean $a(\mathbf{r})$ is a constant which can be taken to be one, and the covariance $R(\mathbf{r}_1, \mathbf{r}_2) = R(\mathbf{r}_1 - \mathbf{r}_2)$. The solution to the Euler equation is simply

$$\mathbf{z}^*(\tau) = (\tau/t)(\mathbf{r} - \boldsymbol{\rho}') \tag{38}$$

In view of (38) and (23), (27) becomes

$$K_1(t, \mathbf{r}, \boldsymbol{\rho}') \sim \exp\left\{ikt - \frac{1}{2}\beta^2 \int_0^t \int_0^t \hat{R}\left[\frac{\tau_1 - \tau_2}{t}(\mathbf{r} - \boldsymbol{\rho}')\right] d\tau_1\, d\tau_2\right\} \tag{39}$$

By definition (28), we see that $\phi^* = t$, and (30) implies that

$$t^* = \tfrac{1}{2}|\mathbf{r} - \boldsymbol{\rho}'| \tag{40}$$

Upon using (40) in (29) and integrating it out in t, the result (30) reduces to

$$I_1(\mathbf{r}) \sim ik \int_{R_2} A(\boldsymbol{\rho}')A_1^*(\mathbf{r}, \boldsymbol{\rho}') \frac{\exp\{ik|\mathbf{r} - \boldsymbol{\rho}'| + ik\phi(\boldsymbol{\rho}')\}}{4\pi|\mathbf{r} - \boldsymbol{\rho}'|}\, d\boldsymbol{\rho}' \tag{41}$$

where, with t^* given by (40),

$$A_1^*(\mathbf{r}, \boldsymbol{\rho}') = \exp\left\{-\frac{1}{2}\beta^2 \int_0^{t^*} \int_0^{t^*} \hat{R}\left[\frac{\tau_1 - \tau_2}{t^*}(\mathbf{r} - \boldsymbol{\rho}')\right] d\tau_1\, d\tau_2\right\} \tag{42}$$

According to (19), we get

$$\Gamma_1(\mathbf{r}) \sim -\frac{ik}{2\pi}\int_{R_2} \frac{x}{|\mathbf{r} - \boldsymbol{\rho}'|^2} \times \left[1 - \frac{1}{ik|\mathbf{r} - \boldsymbol{\rho}'|} + \frac{1}{ik}\frac{\partial}{\partial r}\ln A_1^*(\mathbf{r}, \boldsymbol{\rho}')\right]A(\boldsymbol{\rho}') \times A_1^*(\mathbf{r}, \boldsymbol{\rho}')\exp\{ik[\phi(\boldsymbol{\rho}') + |\mathbf{r} - \boldsymbol{\rho}'|\}\, d\boldsymbol{\rho}' \tag{43}$$

Similarly, the second and higher moments can be computed. For brevity, we give only the following results for the second and fourth moments, letting $t_j^* = \frac{1}{2}|\mathbf{r}_j - \boldsymbol{\rho}_j'|$:

$$\Gamma_2(\mathbf{r}_1, \mathbf{r}_2) \sim \frac{k_1k_2}{4\pi^2}\int_{R_2}\int_{R_2} \frac{x_1x_2}{|\mathbf{r}_1 - \boldsymbol{\rho}_1'|\,|\mathbf{r}_2 - \boldsymbol{\rho}_2'|}\left[1 - \frac{1}{ik_1|\mathbf{r}_1 - \boldsymbol{\rho}_1'|} + \frac{1}{ik_1}\frac{\partial}{\partial r_1}\ln A_2^*(\mathbf{r}_1, \mathbf{r}_2, \boldsymbol{\rho}_1', \boldsymbol{\rho}_2')\right]\left[1 + \frac{1}{ik_2|\mathbf{r}_2 - \boldsymbol{\rho}_2'|} - \frac{1}{ik_2}\frac{\partial}{\partial r_2}\ln A_2^*(\mathbf{r}_1, \mathbf{r}_2, \boldsymbol{\rho}_1', \boldsymbol{\rho}_2')\right] \times A(\boldsymbol{\rho}_1')A(\boldsymbol{\rho}_2')A_2^*(\mathbf{r}_1, \mathbf{r}_2, \boldsymbol{\rho}_1', \boldsymbol{\rho}_2') \times \exp\{ik_1[\phi(\rho_1') + |\mathbf{r}_1 - \boldsymbol{\rho}_1|] - ik_2[\phi(\boldsymbol{\rho}_2') + |\mathbf{r}_2 - \boldsymbol{\rho}_2'|]\}\, d\boldsymbol{\rho}_1'\, d\boldsymbol{\rho}_2' \tag{44}$$

and

$$\Gamma_4(\mathbf{r}_1, ..., \mathbf{r}_4) \sim \frac{1}{16\pi^4}\int_{R_2}\cdots\int_{R_2}\prod_{j=1}^{4}\left\{i_j\frac{k_jx_j}{|\mathbf{r}_j - \boldsymbol{\rho}_j'|} \times \left[1 + \frac{(-1)^j}{ik_j|\mathbf{r}_j - \boldsymbol{\rho}_j'|} + \frac{i_j}{k_j}\frac{\partial}{\partial r_j}\ln A_4^*(\mathbf{r}_1, ..., \mathbf{r}_4, \boldsymbol{\rho}_1', ..., \boldsymbol{\rho}_4')\right]A(\boldsymbol{\rho}_j') \times A_4^*(\mathbf{r}_1, ..., \mathbf{r}_4, \boldsymbol{\rho}_1', ..., \boldsymbol{\rho}_4')\right\} \times \exp\left\{\sum_{j=1}^{4} i_jk_j[\phi(\boldsymbol{\rho}_j') + |\mathbf{r}_j - \boldsymbol{\rho}_j|]\right\} d\boldsymbol{\rho}_1'\cdots d\boldsymbol{\rho}_4' \tag{45}$$

where our convention (15) is applied to i_j, k_j, and β_j, and for $m = 2, 4$

$$A_m^*(\mathbf{r}_1, \ldots, \mathbf{r}_m, \boldsymbol{\rho}_1', \ldots, \boldsymbol{\rho}_m')$$

$$= \exp\left\{-\frac{\beta^2}{2}\sum_{j=1}^{m}\int_0^{t_j^*}\int_0^{t_j^*}\hat{R}\left[\frac{\tau_1 - \tau_2}{t_j^*}(\mathbf{r}_j - \boldsymbol{\rho}_j')\right]d\tau_1\,d\tau_2\right.$$

$$\left.+\frac{1}{2}\sum_{\substack{j\neq l\\ j,l=1}}^{m}\beta_j\beta_l\int_0^{t_j^*}\int_0^{t_l^*}\hat{R}\left[\frac{\tau_1}{t_j^*}(\mathbf{r}_j - \boldsymbol{\rho}_j') - \frac{\tau_2}{t_l^*}(\mathbf{r}_l - \boldsymbol{\rho}_l')\right]d\tau_1\,d\tau_2\right\} \tag{46}$$

We note that, when $A_1^* \equiv 1$, (43) reduces to the case of propagation in a homogeneous medium.[10] The effects of random fluctuations on the moments at high frequencies are completely described by the functions A_j^* appearing in Eq. (46). To simplify the results further, let us specify the aperture field as follows:

$$u(0, \boldsymbol{\rho}) = A_0 \exp\left\{\frac{1}{2}\left(\frac{ik}{R_0} - \frac{1}{a_0^2}\right)\rho^2\right\} \tag{47}$$

which is the profile of a Gaussian beam with a maximum amplitude A_0, the initial effective beam radius α_0, and the radius of curvature of the focused wave front R_0. By a comparison with (2), we get

$$A(\boldsymbol{\rho}) = A_0 \exp(\rho^2/2\alpha_0^2) \tag{48}$$

$$\phi(\boldsymbol{\rho}) = \rho^2/2R_0 \tag{49}$$

In view of (43), $\phi_1(\boldsymbol{\rho}) = |\mathbf{r} - \boldsymbol{\rho}|$, and noting (39), Eq. (33) for the stationary point $\boldsymbol{\rho}^*$ reads

$$(1/R_0)\boldsymbol{\rho}^* - |\mathbf{r} - \boldsymbol{\rho}^*|^{-1}(\boldsymbol{\rho} - \boldsymbol{\rho}^*) = 0 \tag{50}$$

which can be solved for $\boldsymbol{\rho}^*$ approximately for a large x,

$$\boldsymbol{\rho}^* = (R_0/x)\boldsymbol{\rho} + O(1/x^2), \qquad x \gg R_0, \quad x \gg |\boldsymbol{\rho}| \tag{51}$$

Corresponding to (51), it can be shown easily that

$$D^{1/2}(\mathbf{r}) = \left(\frac{1}{R_0} + \frac{1}{x}\right) + O\left(\frac{1}{x^2}\right) \tag{52}$$

When (51) and (52) are used in (37) with the terms of $O(1/x^2)$ neglected, we obtain

$$\Gamma_1(\mathbf{r}) \sim \frac{R_0 x^2}{\zeta(\mathbf{r})(x + R_0)}\left\{1 - (ik)^{-1}\left[\frac{1}{\zeta(\mathbf{r})} - \frac{1}{2}k^2\epsilon^2\int_0^{\zeta(\mathbf{r})} R(\tau)\,d\tau\right]\right\}$$

$$\times \exp\left\{ik\zeta(\mathbf{r}) + \frac{1}{2}\left(ik - \frac{R_0}{\alpha_0}\right)\frac{\rho^2}{x^2}\right.$$

$$\left.- k^2\epsilon^2\int_0^{\zeta(\mathbf{r})}[2\zeta(\mathbf{r}) - \tau]R(\tau)\,d\tau\right\} \tag{53}$$

where, for simplicity, the random refractive index is assumed to be isotropic and ζ is defined as

$$\zeta(\mathbf{r}) = |\mathbf{r} - (R_0/x)\boldsymbol{\rho}| \tag{54}$$

In a similar fashion, the second moment (44) can be simplified to give

$$\begin{aligned}\Gamma_2(\mathbf{r}_1, \mathbf{r}_2) \sim R_0^2\Bigg\{\prod_{l=1}^{2}\Bigg[1 - (i_l k_l)^{-1} \\ \times \left(\frac{1}{\zeta(\mathbf{r}_l)} + k^2\epsilon^2\int_0^{\zeta(\mathbf{r}_l)/2} R(\zeta(\mathbf{r}_l) - 2\tau)\, d\tau\right)\Bigg] \\ + \frac{1}{2}k^2\epsilon^2\frac{R(0)}{\zeta(\mathbf{r}_l)}\Bigg\}\prod_{l=1}^{2}\frac{x_l^2}{(x_l + R_0)\zeta(\mathbf{r}_l)} \\ \times \exp\Bigg\{i_l k_l \zeta(\mathbf{r}_l) + \frac{1}{2}\left(i_l k_l - \frac{R_0}{\alpha_0}\right)\left(\frac{\rho_l}{x_l}\right)^2 \\ - k_l^2\epsilon^2\int_0^{\zeta(\mathbf{r}_l)}[2\zeta(\mathbf{r}_l) - \tau]R(\tau)\, d\tau\Bigg\}\end{aligned} \tag{55}$$

To compute the intensity I, we let $\mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r}$ in (55) and assume that the wave number k is real to get

$$\begin{aligned}I(\mathbf{r}) = \Gamma_2(\mathbf{r}, \mathbf{r}) \sim \frac{R_0^2 x^4}{(x + R_0)^2\zeta^2(\mathbf{r})}\left\{1 + \frac{1}{k^2}\left[\frac{1}{\zeta(\mathbf{r})} - \frac{k^2\epsilon^2}{2}\int_0^{\zeta(\mathbf{r})} R(\tau)\, d\tau\right]^2\right\} \\ \times \exp\left\{-\frac{R_0}{\alpha_0}\left(\frac{\rho}{x}\right)^2 - \frac{k^2\epsilon^2}{2}\int_0^{\zeta(\mathbf{r})}[2\zeta(\mathbf{r}) - \tau]R(\tau)\, d\tau\right\}\end{aligned} \tag{56}$$

Far along the beam axis, $\boldsymbol{\rho} = 0$ and $x \to \infty$, (53) and (56) yield the following simple expressions:

$$\begin{aligned}\Gamma_1(x) \sim R_0\left[1 - ik\epsilon^2\int_0^\infty R(\tau)\, d\tau\right] \\ \times \exp\left\{\tfrac{1}{4}k^2\epsilon^2\int_0^\infty \tau R(\tau)\, d\tau + ikx - \tfrac{1}{2}k^2\epsilon^2 x\int_0^\infty R(\tau)\, d\tau\right\}\end{aligned} \tag{57}$$

$$\begin{aligned}I(x) \sim R_0^2\left\{1 + k^2\epsilon^4\left[\int_0^\infty R(\tau)\, d\tau\right]^2\right\} \\ \times \exp\left\{\tfrac{1}{2}k^2\epsilon^2\int_0^\infty \tau R(\tau)\, d\tau - k^2\epsilon^2 x\int_0^\infty R(\tau)\, d\tau\right\}\end{aligned} \tag{58}$$

We see clearly that both the mean field and the intensity decay exponentially with a rate proportional to $\int_0^\infty R(\tau)\, d\tau$. In fact we have $I(x) = \Gamma_1(x)\overline{\Gamma}_1(x)$.

Therefore, along the beam axis, fluctuations about the mean value are negligible. The fourth moment Γ_4 can also be computed. The physical significance of our results and comparison with results obtained by other workers will be discussed elsewhere. For example, one can show that, by a Fresnel integral and other approximations, our general results (54) and (56) yield the results obtained in Refs. 3 and 4 by different methods.

4.2. Lens-Like Random Media

For a focusing gas lens with random inhomogeneities, the mean field $a(\mathbf{r})$ is assumed to be of the form

$$a(\mathbf{r}) = 1 - \tfrac{1}{4}q_1{}^2\rho_1{}^2 - \tfrac{1}{4}q_2{}^2\rho_2{}^2 \tag{59}$$

where $\boldsymbol{\rho} = (\rho_1, \rho_2)$; q_1, q_2 are some positive constants; and the correlation function R is assumed to be homogeneous. Corresponding to (59), the Euler equation (24) takes the form

$$\frac{d^2\mathbf{z}}{d\tau^2} + \mathbf{q}\cdot\mathbf{z} = 0, \qquad \mathbf{z}(0) = 0, \qquad \mathbf{z}(t) = \mathbf{r} - \boldsymbol{\rho}' \tag{60}$$

where $\mathbf{q} = (0, q_1{}^2, q_2{}^2)$.

The solutions to (60) are given by

$$z_1{}^* = \frac{\tau}{t}x, \qquad z_{j+1}^* = \frac{\sin q_j\tau}{\sin q_j t}(\rho_j - \rho_j'), \qquad j = 1, 2 \tag{61}$$

Upon using (59) and (61) in (26) and evaluating G_t at $\mathbf{z}^*$, we obtain

$$\begin{aligned} K_1(t, \mathbf{r}, \boldsymbol{\rho}') \sim G_t[\mathbf{z}^*] \int_{C(t,0)} &\exp\left\{\frac{ik}{4}\int_0^t [y(\tau)]^2\, d\tau \right.\\ &\left. - \frac{ik}{4}\int_0^t \sum_{j=1}^{2} q_j{}^2 y_{j+1}^2(\tau)\, d\tau\right\} d_{\mathrm{W}}\mathbf{y} \\ &\times \exp\left\{ikt + \frac{ik}{4}\frac{x^2}{t} + \frac{ik}{4}\right. \\ &\left.\times \int_0^t \sum_{j=1}^{2} q_j{}^2 \frac{\cos 2q_j\tau}{(\sin q_j t)^2}(\rho_j - \rho_j')^2\, d\tau\right\} \end{aligned} \tag{62}$$

where

$$\begin{aligned} G_t[\mathbf{z}^*] = \exp&\left\{-\frac{1}{2}\beta^2\right. \\ &\times \int_0^t\int_0^t R\left\{\frac{\tau_1 - \tau_2}{t}x, \sum_{j=1}^{2}\frac{\sin q_j\tau_1 - \sin q_j\tau_2}{\sin q_j t}(\rho_j - \rho_j')\right\} \\ &\left.\times\, d\tau_1\, d\tau_2\right\} \end{aligned} \tag{63}$$

Letting

$$\lambda_j = (k/4i)q_j^2 \tag{64}$$

the Wiener integral of the exponential quadratic functional in (62) can be evaluated[11] and then simplified to give

$$K_1(t, \mathbf{r}, \boldsymbol{\rho}') \sim G_t[\mathbf{z}^*](4\pi ik^{-1}t^3)^{-1/2} \prod_{j=1}^{2} [\sin(\lambda_j^{1/2}t)]^{-1/2}$$

$$\times \exp\left\{ikt + \frac{1}{4} ik \frac{x^2}{t} + \frac{1}{4} ik \sum_{j=1}^{2} q_j(\cot q_j t)(\rho_j - \rho_j')^2\right\} \tag{65}$$

where the principal branch of each root function is taken. By a comparison with (28), we get

$$\phi_1(t, \mathbf{r}, \rho') = t + \frac{x^2}{4t} + \frac{1}{4} \sum_{j=1}^{2} q_j(\cot q_j t)(\rho_j - \rho_j')^2 \tag{66}$$

and

$$\frac{\partial \phi_1}{\partial t} = 1 - \frac{x^2}{4t^2} - \frac{1}{4} \sum_{j=1}^{2} q_j^2(\csc^2 q_j t)(\rho_j - \rho_j')^2 = 0 \tag{67}$$

It can be shown that, when

$$1 - \frac{x^2}{t^2} - \frac{1}{4} \sum_{j=1}^{2} q_j^2(\rho_j - \rho_j')^2 > 0$$

the transcendental equation (67) has infinitely many positive solutions $t^* = t_1, t_2, ..., t_n, ...$, as functions of $\mathbf{r}$ and $\boldsymbol{\rho}'$. However, the explicit determination of t^* in a closed form becomes impossible. Therefore we shall not carry out the asymptotic evaluation for integrals with respect to t and $\boldsymbol{\rho}'$.

By virtue of (65), (25), and (21), Eq. (19) yields the following result:

$$\Gamma_1(\mathbf{r}) \sim 2ik^{-1} \frac{\partial}{\partial x} \int_{R_2} \int_0^{\infty} \frac{A(\boldsymbol{\rho}')G_t[\mathbf{z}^*]}{(4\pi ik^{-1}t^3)^{1/2} \prod_{j=1}^{2} [\sin(\lambda_j^{1/2}t)]^{1/2}}$$

$$\times \exp\left\{ik\left[\phi(\boldsymbol{\rho}')\right.\right.$$

$$\left.\left. + t + \frac{x^2}{4t} + \frac{1}{4} \sum_{j=1}^{2} q_j(\cot q_j t)(\rho_j - \rho_j')^2\right]\right\} d\boldsymbol{\rho}'\, dt \tag{68}$$

For higher moments, we shall not write down their lengthy expressions, except the second moment Γ_2, which is found to be

$$\Gamma_2(\mathbf{r}_1, \mathbf{r}_2) \sim \frac{1}{4k\pi} \frac{\partial^2}{\partial x_1 \partial x_2} \int_{R_2} \int_{R_2} \int_0^\infty \int_0^\infty G_{t_1 t_2}[\mathbf{z}_1^*, \mathbf{z}_2^*]$$
$$\times \prod_{l=1}^{2} \frac{A(\boldsymbol{\rho}_l')}{t_l^{3/2}[\sin(\lambda_1^{1/2} t_l) \sin(\lambda_2^{1/2} t_l)]^{1/2}}$$
$$\times \exp\left\{ ik(-1)^{l+1} \left[\phi(\boldsymbol{\rho}_l') + t_l + \frac{x_l^2}{4t_l} \right.\right.$$
$$\left.\left. + \frac{1}{4} \sum_{j=1}^{2} q_j (\cot q_j t_l)(\rho_{lj} - \rho_{lj}')^2 \right] \right\}$$
$$\times d\boldsymbol{\rho}_1' \, d\boldsymbol{\rho}_2' \, dt_1 \, dt_2 \tag{69}$$

where k is real and the function

$$G_{t_1 t_2}[\mathbf{z}_1^*, \mathbf{z}_2^*]$$
$$= \exp\left\{ -\frac{1}{2} \beta^2 \left[\sum_{l=1}^{2} \int_0^{t_l} \int_0^{t_l} R\left(\frac{\tau_1 - \tau_2}{t_l} x_l, \right.\right.\right.$$
$$\left. \sum_{j=1}^{2} \frac{\sin q_j \tau_1 - \sin q_j \tau_2}{\sin q_j t_l} (\rho_{lj} - \rho_{lj}')^2 \right) d\tau_1 \, d\tau_2$$
$$+ 2 \int_0^{t_1} \int_0^{t_2} R\left(\frac{\tau_1}{t_1} x_1 - \frac{\tau_2}{t_2} x_2, \right.$$
$$\left.\left.\left. \sum_{j=1}^{2} \frac{\sin q_j \tau_1}{\sin q_j t_1} (\rho_{1j} - \rho_{1j}')^2 - \sum_{j=1}^{2} \frac{\sin q_j \tau_2}{\sin q_j t_2} (\rho_{2j} - \rho_{2j}')^2 \right) d\tau_1 \, d\tau_2 \right] \right\} \tag{70}$$

Although we have been able to obtain the moments by the present approach, the results such as (68) and (69) are too complicated to admit physical interpretation. The effects of random fluctuations on the beam propagation are contained in the functions G_t and $G_{t_1 t_2}$ given, respectively, by (63) and (70), and the like for higher moments. Since the correlation function R is positive definite, these functions give rise to the decay factors for the moments. It is of great physical interest to simplify the results further to reveal how the moments are actually modulated by random inhomogeneities, yet without destroying the validity of this approximation. This is a nontrivial problem which, we hope, will challenge some workers in this field.

APPENDIX: ON THE PARABOLIC EQUATION APPROXIMATION

The parabolic equation approximation to the random reduced wave equation (1) is a deterministic approximation.[6,12] We wish to show that it corresponds to a unidirectional asymptotic expansion of the functional integral representation (9) to the half-space problem (1)–(2). To this end, we substitute (10), where we set $a = 1$, into (9) to get

$$u(\mathbf{r}) = 2ik^{-1}\frac{\partial}{\partial x}\int_{R_2}\int_0^\infty \int_{C(t,x)}\int_{C(t,\rho-\rho')} u_0(\rho') + \exp\left\{ikt + ik\epsilon\int_0^t \mu(z_1(\tau), \mathbf{z}_\perp(\tau) + \rho')\, d\tau + \frac{ik}{4}\int_0^t [\dot{z}_1(\tau)]^2\, d\tau + \frac{ik}{4}\int_0^t [\dot{\mathbf{z}}_\perp(\tau)]^2\, d\tau\right\} d\rho'\, dt\, d_W z_1\, d_W \mathbf{z}_\perp \quad (A.1)$$

where we have split the integral with respect to $\mathbf{z}(\tau)$ into the longitudinal component $z_1(\tau)$ and the transverse component $\mathbf{z}_\perp(\tau)$. Again, we keep $\beta = k\epsilon$ fixed and carry out a stationary phase evaluation of the integral (A.1) with respect to $z_1(\tau)$ only. The stationary path is easily found to be $z_1^* = (\tau/t)x$; when used in (A.1), we obtain

$$u(\mathbf{r}) \sim 2ik^{-1}\frac{\partial}{\partial x}\int_{R_2}\int_0^\infty \frac{u_0(\rho')}{(4\pi ik^{-1}t)^{1/2}}\exp\left\{ik\frac{x^2}{4t} + ikt\right\} \times \left(\int_{C(t,\rho-\rho')} \exp\left\{ik\epsilon\int_0^t \mu\left(\frac{\tau}{t}x, \mathbf{z}_\perp(\tau) + \rho'\right) d\tau + \frac{ik}{4}\int_0^t [\dot{\mathbf{z}}_\perp(\tau)]^2\, d\tau\right\} d_W\mathbf{z}_\perp\right) d\rho'\, dt \quad (A.2)$$

In (A.2), the phase in the t-integration is $[(x^2/4t) + t]$, which is stationary when $t^* = x/2$. With this stationary t^*, (A.2) can be asymptotically reduced to

$$u(\mathbf{r}) \sim (ik)^{-1}(\partial/\partial x)\int_{R_2}\int_{C(t,\rho-\rho')} u_0(\rho') \times \exp\left\{ikx + ik\epsilon\int_0^{x/2} \mu(2\tau, \mathbf{z}_\perp(\tau) + \rho')\, d\tau + \tfrac{1}{4}ik\int_0^{x/2} [\dot{\mathbf{z}}_\perp(\tau)]^2\, d\tau\right\} d\rho'\, d_W\mathbf{z}_\perp \quad (A.3)$$

Letting $\sigma = 2\tau$ in (A.3), it can be rewritten as

$$u(\mathbf{r}) \sim e^{ikx}[v(\mathbf{r}) + (ik)^{-1}\,\partial v(\mathbf{r})/\partial x] \quad (A.4)$$

where

$$v(\mathbf{r}) = \int_{R_2}\int_{C(t,\rho-\rho')} u_0(\rho')\exp\left\{\tfrac{1}{2}ik\epsilon\int_0^x \mu(\sigma, \mathbf{z}_\perp(\sigma) + \rho')\, d\sigma + ik\int_0^x [\dot{\mathbf{z}}(\sigma)]^2\, d\sigma\right\} d\rho'\, d_W\mathbf{z}_\perp \quad (A.5)$$

In view of (A.5), v satisfies the parabolic equation

$$\partial v/\partial x = (i/2k)(\Delta_T + \mu)v$$

and

$$v|_{x=0} = u_0(\boldsymbol{\rho}) \tag{A.6}$$

Since the parabolic equation approximation is given by (A.4) with the term $(ik)^{-1}\,\partial v/\partial x$ neglected on the right-hand side, our result (A.4) constitutes an improved parabolic approximation, though the correction term may be small for large k.

ACKNOWLEDGMENT

The author wishes to thank Prof. Elliott W. Montroll of the University of Rochester for his helpful comments, which led to an improvement of the original manuscript.

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